ELSEVIER
Journal of Geometry and Physics 44 (2002) 279–298
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Bäcklund theorems in three-dimensional de Sitter space and anti-de Sitter space

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Received 13 April 2001; received in revised form 14 March 2002

Abstract

In three-dimensional de Sitter space S_1^3 and anti-de Sitter space H_1^3, we generalize the classical Bäcklund theorem. Moreover, we obtain explicit forms of Bäcklund transformations (BTs) in the Tchebyshev coordinates and investigate the relation of loop group actions and BTs in S_1^3.

MSC: 53A05; 53B30; 58F37

Subj. Class.: Differential geometry; General relativity

Keywords: de Sitter space; Anti-de Sitter space; Bäcklund transformation; Loop group action

1. Introduction

The classical Bäcklund theorem [1,2] studies the transformation of surfaces with constant negative curvature in the Euclidean space E^3 by realizing them as the focal surfaces of a pseudo-spherical line congruence. The integrability theorem says that we can construct a new surface in E^3 with constant negative curvature from a given one by using the Bäcklund transformation (BT in brief).

With the development of the integrable system theory, BT has become an important method to find the solutions of integrable equations, specially soliton equations (see [11–17]). At the same time the geometricians also pay attention to the generalization and development of geometrical content of the Bäcklund theorem [2–9]. In [2], Chern and Terng introduced W-congruence and discussed BT between affine minimal surfaces in affine geometry. In [6] Antonowicz presented an analytic form of the affine BT and constructed some new

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PII: S0393-0440(02)00077-3

affine minimal surfaces in A^3. In [3–5], Tenenblat and Terng considered the generalization of Bäcklund theorem in high dimensional space forms and generalized the sine-Gordon equation and wave equation. In [16,17], the relation of BTs and loop group actions was studied. In Lorentzian space forms, the analogue of Bäcklund theorem was considered in $R^{2,1}$ (see [7–13,18]). It is not clear in the three-dimensional de Sitter space S_1^3 and anti-de Sitter space H_1^3 which are important spaces in physics and used as cosmological models in general relativity. To study, it is an interesting thing both in geometry and in physics.

The aim of this paper is to study the BTs of constant curved surfaces in S_1^3 and H_1^3. Firstly, we discuss the Bäcklund congruences (BCs in brief) between surface in both S_1^3 and H_1^3. Since there are time-like and space-like surfaces in our scope (according to induced metrics are either Riemannian or Lorentzian), and the BCs may be either space-like or time-like, we should separate the BCs into following cases:

1. space-like BC between time-like surface and space-like surface with $K = 1 - \rho^2$,
2. space-like BC between space-like surface and space-like surface with $K = 1 + \rho^2$,
3. space-like BC between time-like surface and time-like surface with $K = 1 + \rho^2$,
4. time-like BC between time-like surface and time-like surface with $K = 1 + \rho^2$,

where K is the Gaussian curvature of surface and $\rho > 0$ is a constant.

By using Tchebyshev coordinates for constant curved surfaces (Lemmas 3.7 and 3.8), when the ambient space is S_1^3, the Gauss–Codazzi equations of the surfaces are the following:

1. sine-Laplace equation
$$\alpha_{xx} + \alpha_{yy} = (\rho^2 - 1)\sin\alpha, \tag{1.1}$$
and sinh-Laplace equation
$$\alpha_{xx} + \alpha_{yy} = (\rho^2 - 1)\sinh\alpha, \tag{1.2}$$
2. sine-Gordon equation
$$\alpha_{xx} - \alpha_{yy} = -(\rho^2 + 1)\sin\alpha, \tag{1.3}$$
and sinh-Gordon equation
$$\alpha_{xx} - \alpha_{yy} = -(\rho^2 + 1)\sinh\alpha, \tag{1.4}$$
3. cosh-Gordon equation
$$\alpha_{xy} + (\rho^2 + 1)\cosh\alpha = 0. \tag{1.5}$$

The corresponding BTs of BCs (1.2)–(1.4) are similar to the classical BT, but the corresponding BT of BC 1 is a transformation between solutions of sine-Laplace equation and sinh-Laplace equation in general, and which includes the BT between solutions of Laplace equation when $\rho = 1$.

The paper is organized as follows: in Section 2 we firstly review the moving frame method for immersed surfaces in S_1^3. Afterwards we generalize the classical Bäcklund theorem in S_1^3 and give the four kinds of BCs. In Section 3 we give the explicit forms of BTs in the Tchebyshev coordinate in S_1^3. In Sections 4 and 5 we shall discuss the relation of loop group actions and BTs in S_1^3. In Section 6, we study the parallel results such as the Bäcklund theorems, in H_1^3. Throughout this paper, we use the summation convention. We assume that $\tau, l \in (0, \pi)$ and $\rho > 0$ are constants and M denotes an immersion without umbilic point in S_1^3 (or H_1^3).

2. The frame method for surfaces and Bäcklund theorems in S_1^3

Let L^4 denote the four-dimensional Minkowski space endowed with linear coordinates (X_0, X_1, X_2, X_3) and the scalar product $\langle\,,\,\rangle$ given by $-X_0^2 + X_1^2 + X_2^2 + X_3^2$. The three-dimensional de Sitter space S_1^3 of constant sectional curvature 1 is defined as the following hyper-quadric in L^4

$$S_1^3 = \{X \in L^4 | \langle X, X\rangle = 1\}.$$

Let M be a simply connected domain and $f : M \to S_1^3 \subset L^4$ an immersion, we choose a local orthonormal frame $\{e_0, e_1, e_2, e_3\}$ such that $e_0 = f$ and $\langle e_0, e_0\rangle = 1$, where e_1, e_2 are tangent vectors and e_3 is normal to M in S_1^3. Suppose e_3 is either space-like or time-like vector. Define the dual coframe $\{\omega^1, \omega^2\}$ of $\{e_1, e_2\}$ by $\omega^i(e_j) = \delta^i_j$ $(i, j = 1, 2)$, then the fundamental equations of M are

$$\begin{aligned} &\mathrm{d}e_0 = \mathrm{d}f = \omega^i e_i, \qquad \mathrm{d}e_1 = -\epsilon_1\omega^1 e_0 + \omega_1^2 e_2 + \omega_1^3 e_3 \quad \epsilon_i\omega_i^j + \epsilon_j\omega_j^i = 0,\\ &\mathrm{d}e_2 = -\epsilon_2\omega^2 e_0 + \omega_2^1 e_1 + \omega_2^3 e_3 \quad (1 \le i \le 2, 1 \le j \le 3), \qquad \mathrm{d}e_3 = \omega_3^i e_i, \end{aligned} \tag{2.1}$$

where ω_1^2 is the connection 1-form, ω_1^3 and ω_2^3 are the second fundamental form of the immersion, and $\epsilon_j = \langle e_j, e_j\rangle = 1$ or -1 $(1 \le j \le 3)$ according to whether e_j is either space-like or time-like.

From $\mathrm{d}^2 e_j = 0$ $(0 \le j \le 3)$, one can obtain the structural equations:

$$\mathrm{d}\omega^1 = \omega^2 \wedge \omega_2^1, \qquad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2, \qquad \epsilon_1\omega_3^1 \wedge \omega^1 + \epsilon_2\omega_3^2 \wedge \omega^2 = 0, \tag{2.2}$$

and

$$\mathrm{d}\omega_1^2 = -\epsilon_1\omega^1 \wedge \omega^2 + \omega_1^3 \wedge \omega_3^2 \quad \text{(Gauss equation)} \tag{2.3}$$

$$\mathrm{d}\omega_3^1 + \omega_2^1 \wedge \omega_3^2 = 0, \qquad \mathrm{d}\omega_3^2 + \omega_1^2 \wedge \omega_3^1 = 0 \quad \text{(Codazzi equation)} \tag{2.4}$$

The two fundamental forms of M are

$$I = \epsilon_1(\omega^1)^2 + \epsilon_2(\omega^2)^2, \qquad II = -\langle \mathrm{d}f, e_3\rangle = -\epsilon_1\omega^1\omega_3^1 - \epsilon_2\omega^2\omega_3^2. \tag{2.5}$$

The eigenvalues k_1 and k_2 of the $II \cdot I^{-1}$ are called the principal curvatures of M.

If e_3 is time-like, the surface is called space-like surface. In this case $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$, and we have

$$\omega_1^2 + \omega_2^1 = 0, \qquad \omega_1^3 = \omega_3^1, \qquad \omega_2^3 = \omega_3^2. \tag{2.6}$$

If e_3 is space-like, the surface is called time-like surface and we may choose $\langle e_2, e_2\rangle = -1$. Then $\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ and

$$\omega_1^2 = \omega_2^1, \qquad \omega_1^3 + \omega_3^1 = 0, \qquad \omega_2^3 = \omega_3^2. \tag{2.7}$$

From (2.2), we get for both space-like and time-like surfaces

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0. \tag{2.8}$$

By Cartan's lemma, one may set $\omega_j^3 = h_{ij}\omega^i$, $h_{ij} = h_{ji}$ $(1 \le i, j \le 2)$. Then for the surface (either space-like or time-like) the Gaussian curvature is defined by $\mathrm{d}\omega_1^2 = -\epsilon_1 K \omega^1 \wedge \omega^2$, and the Gauss equation can be rewritten as

$$K = 1 - \det(h_{ij}). \tag{2.9}$$

It is easily verified that for space-like surface $K = -k_1k_2$, for time-like surface $K = k_1k_2$.

Naturally, as a generalization of classical pseudo-spherical line congruence, we introduce four kinds of BCs in S_1^3 (or H_1^3). In the following, "*line*" means geodesic of target space S_1^3 (or H_1^3).

Definition 2.1. A line congruence between two surfaces M and M^* in S_1^3 (or H_1^3) is a diffeomorphism $l : M \to M^*$ such that for each $P \in M$, the line joining P and $P^* = l(P)$ is a common tangent line for M and M^*. The line congruence l is called a space-like Bäcklund congruence (SBC in brief) (or time-like Bäcklund congruence (TBC in brief)) if

(i) the length of line segment $\overline{PP^*} = l$ is a non-zero constant independent of P,
(ii) the tangent vector of line $\overline{PP^*}$ is space-like (or time-like) vector,
(iii) $\langle n_P, n_P^*\rangle = c$ is a non-zero constant independent of P, where n_P and n_P^* are normal to M and M^*, respectively.

In fact the above BCs in S_1^3 could be separated into the following four cases:

(1) SBC l_1 between time-like surface and space-like surface,
(2) SBC l_2 between space-like surface and space-like surface,
(3) SBC l_3 between time-like surface and time-like surface,
(4) TBC l_4 between time-like surface and time-like surface.

Now we discuss an analogue of classical Bäcklund theorem in S_1^3.

Theorem 2.2. *Let M and M^* be two immersed surfaces in S_1^3. Let $l_i : M \to M^*$ $(1 \le i \le 4)$ be one of the above BCs as in Definition 2.1. Then M and M^* have the same constant Gaussian curvature K, where in* (1) $K = 1 - (\cosh^2\tau / \sin^2 l)$ *and* $c = \sinh\tau$; (2) $K = 1 + (\sinh^2\tau/\sin^2 l)$ *and* $c = -\cosh\tau$; (3) $K = 1 + (\sinh^2\tau/\sin^2 l)$ *and* $c = \cosh\tau$; *and* (4) $K = 1 + (\sin^2\tau/\sinh^2 l)$ *and* $c = \cos\tau$.

Proof.

Case 1. Let $e_0 : M \to S_1^3$ and $e_0^* : M^* \to S_1^3$ be immersed time-like and space-like surfaces, respectively. By the definition, let e_1 (or e_1^*) be the space-like unit tangent vector field of M (or M^*) which is tangent of line congruence l_1, then there exist local orthonormal frames $\{e_i\}$ and $\{e_i^*\}$ $(0 \le i \le 3)$ of M and M^*, respectively, such that

$$\text{SBC } l_1 : \begin{cases} e_0^* = \cos l e_0 + \sin l e_1 \\ e_1^* = -\sin l e_0 + \cos l e_1 \\ e_2^* = \sinh \tau e_2 + \cosh \tau e_3 \\ e_3^* = \cosh \tau e_2 + \sinh \tau e_3, \end{cases} \tag{2.10}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, and $\langle e_1^*, e_1^*\rangle = \langle e_2^*, e_2^*\rangle = -\langle e_3^*, e_3^*\rangle = 1$.

Taking the exterior derivative of e_0^*, we get

$$\mathrm{d}e_0^* = -\omega^1 \sin l e_0 + \omega^1 \cos l e_1 + (\omega^2 \cos l + \omega_1^2 \sin l) e_2 + \omega_1^3 \sin l e_3. \tag{2.11}$$

On the other hand, letting $\{\omega^{*1}, \omega^{*2}\}$ be the dual coframe of $\{e_1^*, e_2^*\}$, we have

$$\begin{aligned} \mathrm{d}e_0^* &= \omega^{*1} e_1^* + \omega^{*2} e_2^* \\ &= -\omega^{*1} \sin l e_0 + \omega^{*1} \cos l e_1 + \omega^{*2} \sinh \tau e_2 + \omega^{*2} \cosh \tau e_3. \end{aligned} \tag{2.12}$$

Comparing coefficients of e_1, e_2, e_3 in (2.11) and (2.12), we have

$$\omega^{*1} = \omega^1, \qquad \omega^{*2} \sinh \tau = \omega^2 \cos l + \omega_1^2 \sin l, \qquad \omega^{*2} \cosh \tau = \omega_1^3 \sin l. \tag{2.13}$$

This gives

$$\omega^2 \cos l + \omega_1^2 \sin l = \omega_1^3 \sin l \tanh \tau. \tag{2.14}$$

By using (2.10), we have

$$\omega_1^{*3} = -\langle e_3^*, \mathrm{d}e_1^*\rangle = -\frac{\cosh \tau}{\sin l}\omega^2, \qquad \omega_2^{*3} = -\langle e_3^*, \mathrm{d}e_2^*\rangle = \omega_2^3. \tag{2.15}$$

By (2.15), (2.8) and (2.13), we have

$$\omega_1^{*3} \wedge \omega_2^{*3} = -\frac{\cosh \tau}{\sin l}\omega^2 \wedge \omega_2^3 = \frac{\cosh \tau}{\sin l}\omega^1 \wedge \omega_1^3 = \frac{\cosh^2 \tau}{\sin^2 l}\omega^{*1} \wedge \omega^{*2}.$$

Now the Gauss equation (2.4) implies that $K^* = 1 - (\cosh^2 \tau / \sin^2 l)$. Note that

$$e_0 = \cos l e_0^* - \sin l e_1^*.$$

By a similar calculation, we know that M also has Gaussian curvature $K = 1 - (\cosh^2 \tau / \sin^2 l)$. This proves the first case of the theorem.

Analogous with Case 1, we may prove Cases 2 and 3. Here we need to notice that the corresponding orthonormal frames are the following:

Case 2.

$$\text{SBC } l_2 : \begin{cases} e_0^* = \cos l e_0 + \sin l e_1 \\ e_1^* = -\sin l e_0 + \cos l e_1 \\ e_2^* = \cosh \tau e_2 + \sinh \tau e_3 \\ e_3^* = \sinh \tau e_2 + \cosh \tau e_3, \end{cases} \tag{2.16}$$

where $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = 1$, and $\langle e_1^*, e_1^*\rangle = \langle e_2^*, e_2^*\rangle = -\langle e_3^*, e_3^*\rangle = 1$.

Case 3.

$$\text{SBC } l_3 : \begin{cases} e_0^* = \cos l e_0 + \sin l e_1 \\ e_1^* = -\sin l e_0 + \cos l e_1 \\ e_2^* = \cosh \tau e_2 + \sinh \tau e_3 \\ e_3^* = \sinh \tau e_2 + \cosh \tau e_3, \end{cases} \tag{2.17}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, and $\langle e_1^*, e_1^*\rangle = -\langle e_2^*, e_2^*\rangle = \langle e_3^*, e_3^*\rangle = 1$.

Case 4. Let $e_0 : M \to S_1^3$ and $e_0^* : M^* \to S_1^3$ be two immersed time-like surfaces. Let e_2 (or e_2^*) be the time-like unit tangent vector field of M (or M^*) which is tangent of line congruence l_4, then there exist local orthonormal frames $\{e_i\}$ and $\{e_i^*\}$ $(0 \le i \le 3)$ of M and M^*, respectively, such that

$$\text{TBC } l_4 : \begin{cases} e_0^* = \cosh l e_0 + \sinh l e_2 \\ e_1^* = \cos \tau e_1 + \sin \tau e_3 \\ e_2^* = \sinh l e_0 + \cosh l e_2 \\ e_3^* = -\sin \tau e_1 + \cos \tau e_3, \end{cases} \tag{2.18}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, and $\langle e_1^*, e_1^*\rangle = -\langle e_2^*, e_2^*\rangle = \langle e_3^*, e_3^*\rangle = 1$.

Taking the exterior derivative of (2.18) we have

$$\omega^{*2} = \omega^2, \qquad \omega^{*1} \cos \tau = \omega^1 \cosh l + \omega_2^1 \sinh l, \qquad \omega^{*1} \sin \tau = \omega_2^3 \sinh l, \tag{2.19}$$

and

$$\omega^1 \cosh l + \omega_2^1 \sinh l = \omega_2^3 \sinh l \cot \tau. \tag{2.20}$$

By using (2.18) and (2.19), we have

$$\omega_1^{*3} = \langle e_3^*, \mathrm{d}e_1^*\rangle = \omega_1^3, \qquad \omega_2^{*3} = \langle e_3^*, \mathrm{d}e_2^*\rangle = \frac{\sin \tau}{\sinh l}\omega^1. \tag{2.21}$$

Then

$$\omega_1^{*3} \wedge \omega_2^{*3} = \frac{\sin \tau}{\sinh l}\omega_1^3 \wedge \omega^1 = \frac{\sin \tau}{\sinh l}\omega^2 \wedge \omega_2^3 = -\frac{\sin^2 \tau}{\sinh^2 l}\omega^{*1} \wedge \omega^{*2}.$$

So $K^* = 1 + (\sin^2\tau / \sinh^2 l)$. Similarly, we have $K = 1 + (\sin^2\tau / \sinh^2 l)$.

3. BTs in S_1^3

3.1. BTs for surfaces in S_1^3

In this section, we shall discuss BTs, i.e., the existence of BCs. From the proof of Theorem 2.2, we know the existence of BCs is equivalent to the existence of space-like unit vector field e_1. Let $e_0 : M \to S_1^3$ be an immersed time-like surface. From (2.14) we consider the following differential system of space-like unit vector field e_1 in Case 1

$$\langle \mathrm{d}e_0, e_2\rangle \cos l + \langle \mathrm{d}e_1, e_2\rangle \sin l + \langle \mathrm{d}e_1, e_3\rangle \sin l \tanh \tau = 0.$$

Denote

$$\eta = \omega_1^3 \sin l \tanh \tau - \omega^2 \cos l - \omega_1^2 \sin l.$$

Then the existence of e_1 is equivalent to that $\eta = 0$ is completely integrable. Since

$$\begin{aligned}\mathrm{d}\eta &= \omega_1^2 \wedge \omega_2^3 \sin l \tanh \tau - \omega^1 \wedge \omega_1^2 \cos l - \mathrm{d}\omega_1^2 \sin l \\ &= \frac{\sin l}{\cosh^2 \tau}\left(K - 1 + \frac{\cosh^2 \tau}{\sin^2 l}\right)\omega^1 \wedge \omega^2 \mod \eta.\end{aligned}$$

Hence $\mathrm{d}\eta \equiv 0 \ (\mathrm{mod}\ \eta)$ if and only if $K = 1 - (\cosh^2\tau / \sin^2 l)$. By the Frobenius theorem, we have

Theorem 3.1. *Suppose M is an immersed time-like surface with* $K = 1 - (\cosh^2\tau / \sin^2 l)$ *in* S_1^3. *Given any unit space-like vector* $v_0 \in T_{p_0}M$, $p_0 \in M$, *which is not a principal direction. Then there exist a unique space-like surface* M^* *with K and the above SBC* l_1 *such that* $l_1(p_0) = \cos l p_0 + \sin l v_0$.

Definition 3.2. Eq. (2.14), $\omega^2 \cos l + \omega_1^2 \sin l = \omega_1^3 \sin l \tanh \tau$, is called the BT between time-like surface and space-like surface in S_1^3.

Similar to Case 1, we also have the following existence theorems to the other cases.

Theorem 3.3. *Suppose M is an immersed space-like* (*or time-like*) *surface with* $K = 1 + (\sinh^2\tau / \sin^2 l)$ *in* S_1^3. *Given any unit space-like vector* $v_0 \in T_{p_0}M$, $p_0 \in M$, *which is not a principal direction. Then there exist a unique space-like* (*or time-like*) *surface* M^* *with K and the above SBC* l_2 (*or* l_3) *such that* $l_2(p_0)$ (*or* $l_3(p_0)$) $= \cos l p_0 + \sin l v_0$.

Definition 3.4. The equation

$$\omega^2 \cos l + \omega_1^2 \sin l = \omega_1^3 \sin l \coth \tau \tag{3.1}$$

is called the BT of between space-like (or time-like) surfaces in S_1^3.

Theorem 3.5. *Suppose M is an immersed time-like surface with* $K = 1 + (\sin^2\tau / \sinh^2 l)$ *in* S_1^3. *Given any unit time-like vector* $v_0 \in T_{p_0}M$, $p_0 \in M$, *which is not a principal direction. Then there exist a unique time-like surface* M^* *with K and the above TBC* l_4 *such that* $l_4(p_0) = \cosh l p_0 + \sinh l v_0$.

Definition 3.6. Eq. (2.20), $\omega^1 \cosh l + \omega_2^1 \sinh l = \omega_2^3 \sinh l \cot \tau$, is called the BT of between time-like surfaces in S_1^3.

3.2. BTs in the Tchebyshev coordinates

In the following, we give the explicit forms of BTs in the Tchebyshev coordinates. In S_1^3 we may set up the Tchebyshev coordinates for surfaces analogous with them in $R^{2,1}$ [19,20].

Lemma 3.7. *Suppose M is an immersed surfaces of S_1^3 with constant curvature $K = 1+\rho^2$, where $\rho > 0$ is a constant.*

(1) *If M is space-like, then there exists a local coordinate system (x, y) such that*

$$I = \cos^2\frac{\alpha}{2}\,\mathrm{d}x^2 + \sin^2\frac{\alpha}{2}\,\mathrm{d}y^2, \qquad II = -\rho\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(\mathrm{d}x^2 - \mathrm{d}y^2), \tag{3.2}$$

and α satisfies the equation

$$\alpha_{xx} - \alpha_{yy} = -(1+\rho^2)\sin\alpha. \tag{3.3}$$

(2) *If M is time-like and the principal curvatures k_1 and k_2 are real, then there exists a local coordinate system (x, y) such that*

$$I = \cosh^2\frac{\alpha}{2}\,\mathrm{d}x^2 - \sinh^2\frac{\alpha}{2}\,\mathrm{d}y^2, \qquad II = \rho\cosh\frac{\alpha}{2}\sinh\frac{\alpha}{2}(\mathrm{d}x^2 - \mathrm{d}y^2), \tag{3.4}$$

and α satisfies the equation

$$\alpha_{xx} - \alpha_{yy} = -(1+\rho^2)\sinh\alpha. \tag{3.5}$$

(3) *If M is time-like and the principal curvatures k_1 and k_2 are imaginary, then there exists a local coordinate system (x, y) such that*

$$I = \mathrm{d}x^2 + 2\sinh\alpha\,\mathrm{d}x\,\mathrm{d}y - \mathrm{d}y^2, \qquad II = 2\rho\cosh\alpha\,\mathrm{d}x\,\mathrm{d}y, \tag{3.6}$$

and α satisfies the equation

$$\alpha_{xy} + (1+\rho^2)\cosh\alpha = 0. \tag{3.7}$$

With time-like surfaces of positive curvature, an important case for which the principal curvatures are imaginary, is often missed in some previous papers on BT. In recent paper [20], the case has been taken up and studied in detail in $R^{2,1}$.

Lemma 3.8. *Suppose M is an immersed surface of S_1^3 with constant curvature $K = 1-\rho^2$, where $\rho > 0$ is a constant.*

(1) *If M is space-like, then there exists a local coordinate system (x, y) such that*

$$I = \cosh^2\frac{\alpha}{2}\,\mathrm{d}x^2 + \sinh^2\frac{\alpha}{2}\,\mathrm{d}y^2, \qquad II = -\rho\cosh\frac{\alpha}{2}\sinh\frac{\alpha}{2}(\mathrm{d}x^2+\mathrm{d}y^2), \tag{3.8}$$

and α satisfies the equation

$$\alpha_{xx} + \alpha_{yy} = (\rho^2 - 1)\sinh\alpha. \tag{3.9}$$

(2) *If M is time-like, then there exists a local coordinate system* (x, y) *such that*

$$I = \cos^2\frac{\alpha}{2}\,dx^2 - \sin^2\frac{\alpha}{2}\,dy^2, \qquad II = \rho\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(dx^2 + dy^2), \tag{3.10}$$

and α *satisfies the equation*

$$\alpha_{xx} + \alpha_{yy} = (\rho^2 - 1)\sin\alpha. \tag{3.11}$$

Now we shall consider the explicit forms of BTs, i.e., Eqs. (2.14), (2.20) and (3.1) in the Tchebyshev coordinates.

Theorem 3.9. *Let M and M^* be immersed time-like surface and space-like surface, respectively in S^3_1. Let $l_1 : M \to M^*$ be the above SBC l_1. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_1,*
(2) *The BT between Eqs. (3.9) and (3.11) is*

$$\frac{1}{2}\sin l(\alpha_x - \tilde{\alpha}_y) = \cos l\cos\frac{\tilde{\alpha}}{2}\sinh\frac{\alpha}{2} + \sinh\tau\sin\frac{\tilde{\alpha}}{2}\cosh\frac{\alpha}{2},$$
$$\frac{1}{2}\sin l(\alpha_y + \tilde{\alpha}_x) = -\cos l\sin\frac{\tilde{\alpha}}{2}\cosh\frac{\alpha}{2} + \sinh\tau\cos\frac{\tilde{\alpha}}{2}\sinh\frac{\alpha}{2}, \tag{3.12}$$

where $\rho = \cosh\tau/\sin l$, α and $\tilde{\alpha}$ satisfy Eqs. (3.9) and (3.11), respectively.

Proof. Suppose $e_0 = f : M \to S^3_1$ is an immersed time-like surface with $K = 1 - (\cosh^2\tau/\sin^2 l)$ covered by the Tchebyshev coordinate (x, y). By Lemma 3.8, we may choose the right orthonormal frame field $\{e_0, h_1, h_2, e_3\}$, where $h_1 = (1/\cos\frac{\tilde{\alpha}}{2})(\partial/\partial x)$, $h_2 = (1/\sin\frac{\tilde{\alpha}}{2})(\partial/\partial y)$ and $\langle h_1, h_1\rangle = 1$, $\langle h_2, h_2\rangle = -1$. Let $\{\eta^1, \eta^2\}$ be the dual coframe of $\{h_1, h_2\}$, and η_i^j be the corresponding connection 1-forms. Then we have

$$\eta^1 = \cos\frac{\tilde{\alpha}}{2}\,dx, \qquad \eta^2 = \sin\frac{\tilde{\alpha}}{2}\,dy, \qquad \eta_1^2 = \tfrac{1}{2}(-\tilde{\alpha}_y\,dx + \tilde{\alpha}_x\,dy),$$
$$\eta_1^3 = \rho\sin\frac{\tilde{\alpha}}{2}\,dx = -\eta_3^1, \qquad \eta_2^3 = \rho\cos\frac{\tilde{\alpha}}{2}\,dy = \eta_3^2,$$

where $\rho = \cosh\tau/\sin l$ is a constant.

Use the same notation in the proof of Theorem 2.2 and suppose

$$e_1 = \cosh\frac{\alpha}{2}h_1 + \sinh\frac{\alpha}{2}h_2, \qquad e_2 = \sinh\frac{\alpha}{2}h_1 + \cosh\frac{\alpha}{2}h_2, \tag{3.13}$$

where e_1 is the SBC direction. By a direct calculation, we have

$$\omega^1 = \cosh\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\,dx + \sinh\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\,dy,$$
$$\omega^2 = -\sinh\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\,dx - \cosh\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\,dy, \qquad \omega_1^2 = \eta_1^2 + \tfrac{1}{2}\,d\alpha,$$
$$\omega_1^3 = \rho\left(\cosh\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\,dx - \sinh\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\,dy\right),$$
$$\omega_2^3 = \rho\left(\sinh\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\,dx - \cosh\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\,dy\right). \tag{3.14}$$

Using (2.13), the fundamental forms of M^* are

$$I^* = (\omega^{*1})^2 + (\omega^{*2})^2 = (\omega^1)^2 + \left(\frac{\sin l}{\cosh\tau}\omega_1^3\right)^2 = \cosh^2\frac{\alpha}{2}\,dx^2 + \sinh^2\frac{\alpha}{2}\,dy^2,$$

$$II^* = -\omega^{*1}\omega_1^{*3} - \omega^{*2}\omega_2^{*3} = \frac{\cosh\tau}{\sin l}\omega^1\omega^2 - \frac{\sin l}{\cosh\tau}\omega_1^3\omega_2^3$$
$$= -\rho\cosh\frac{\alpha}{2}\sinh\frac{\alpha}{2}(dx^2 + dy^2).$$

By Lemma 3.8, we know (1) holds. Substituting (3.14) in (2.14), comparing the coefficients of dx, dy in (2.14), we get the BT (3.12). □

Similarly to the other cases, we also have the following theorems:

Theorem 3.10. *Let M and M^* be two immersed space-like surfaces in S_1^3. Let $l_2 : M \to M^*$ be the above SBC l_2. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_2,*
(2) *The BT between Eqs. (3.3) and (3.3) is*

$$\frac{1}{2}\sin l(\tilde{\alpha}_x + \alpha_y) = \cos l\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2} + \cosh\tau\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2},$$
$$\frac{1}{2}\sin l(\tilde{\alpha}_y + \alpha_x) = -\cos l\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2} - \cosh\tau\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}, \quad (3.15)$$

where $\rho = \sinh\tau/\sin l$, α and $\tilde{\alpha}$ satisfy Eq. (3.3).

Theorem 3.11. *Let M and M^* be two immersed time-like surfaces with real principal curvatures in S_1^3. Let $l_3 : M \to M^*$ be the above SBC l_3. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_3,*
(2) *The BT between Eqs. (3.5) and (3.5) is*

$$\frac{1}{2}\sin l(\tilde{\alpha}_x + \alpha_y) = \cos l\cosh\frac{\alpha}{2}\sinh\frac{\tilde{\alpha}}{2} + \cosh\tau\sinh\frac{\alpha}{2}\cosh\frac{\tilde{\alpha}}{2},$$
$$\frac{1}{2}\sin l(\tilde{\alpha}_y + \alpha_x) = -\cos l\sinh\frac{\alpha}{2}\cosh\frac{\tilde{\alpha}}{2} - \cosh\tau\cosh\frac{\alpha}{2}\sinh\frac{\tilde{\alpha}}{2}, \quad (3.16)$$

where $\rho = \sinh\tau/\sin l$, α and $\tilde{\alpha}$ satisfy Eq. (3.5).

Theorem 3.12. *Let M and M^* be two immersed time-like surfaces with real principal curvatures in S_1^3. Let $l_4 : M \to M^*$ be the above TBC l_4. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_4.*
(2) *The BT between Eqs. (3.5) and (3.5) is*

$$\frac{1}{2}\sinh l(\alpha_x + \tilde{\alpha}_y) = \cosh l\cosh\frac{\alpha}{2}\sinh\frac{\tilde{\alpha}}{2} + \cos\tau\sinh\frac{\alpha}{2}\cosh\frac{\tilde{\alpha}}{2},$$
$$\frac{1}{2}\sinh l(\alpha_y + \tilde{\alpha}_x) = -\cosh l\sinh\frac{\alpha}{2}\cosh\frac{\tilde{\alpha}}{2} - \cos\tau\cosh\frac{\alpha}{2}\sinh\frac{\tilde{\alpha}}{2}, \quad (3.17)$$

where $\rho = \sin\tau/\sinh l$, α and $\tilde{\alpha}$ satisfy Eq. (3.5).

Now we discuss the BT between time-like surfaces of positive constant curvature and imaginary principal curvatures in S_1^3.

Theorem 3.13. *Let M and M^* be two immersed time-like surfaces with imaginary principal curvatures in S_1^3. Let $l_3 : M \to M^*$ be the above TBC l_3. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_3.*
(2) *The BT between Eqs. (3.7) and (3.7) is*

$$\frac{\alpha_x+\tilde{\alpha}_x}{2}=\frac{\cosh\tau+\cos l}{\sin l}\sinh\frac{\alpha-\tilde{\alpha}}{2},\qquad \frac{\alpha_y-\tilde{\alpha}_y}{2}=\frac{\cos l-\cosh\tau}{\sin l}\cosh\frac{\alpha+\tilde{\alpha}}{2}, \tag{3.18}$$

where $\rho = \sinh\tau/\sin l$, α and $\tilde{\alpha}$ satisfy Eq.(3.7).

Theorem 3.14. *Let M and M^* be two immersed time-like surfaces with imaginary principal curvatures in S_1^3. Let $l_3 : M \to M^*$ be the above TBC l_4. Then*

(1) *The Tchebyshev coordinates of M and M^* correspond under l_4,*
(2) *The BT between Eqs. (3.7) and (3.7) is*

$$\frac{\alpha_x-\tilde{\alpha}_x}{2}=\frac{\cos\tau-\cosh l}{\sinh l}\cosh\frac{\alpha+\tilde{\alpha}}{2},\qquad \frac{\alpha_y+\tilde{\alpha}_y}{2}=\frac{\cosh l+\cos\tau}{\sinh l}\sinh\frac{\alpha-\tilde{\alpha}}{2}, \tag{3.19}$$

where $\rho = \sin\tau/\sinh l$, α and $\tilde{\alpha}$ satisfy Eq.(3.7).

Proof. Suppose $e_0 = f : M \to S_1^3$ is an immersed time-like surface with $K = 1 + (\sin^2\tau/\sinh^2 l)$ covered by the Tchebyshev coordinate (x, y). By Lemma 3.7, we may choose the right orthonormal frame field $\{e_0, h_1, h_2, e_3\}$, where $\langle h_1, h_1\rangle = 1$, $\langle h_2, h_2\rangle = -1$. Let $\{\eta^1, \eta^2\}$ be the dual coframe of $\{h_1, h_2\}$, and η_i^j be the corresponding connection 1-forms. Then we have

$$\eta^1 = dx + \sinh\tilde{\alpha}\,dy,\qquad \eta^2 = \cosh\tilde{\alpha}\,dy,\qquad \eta_1^2 = -\tilde{\alpha}_x\,dx = \eta_2^1,$$
$$\eta_1^3 = \rho\cosh\tilde{\alpha}\,dy = -\eta_3^1,\qquad \eta_2^3 = dx - \rho\sinh\tilde{\alpha}\,dy = \eta_3^2,$$

where $\rho = \sin\tau/\sinh l$ is a constant.

Use the same notation in the proof of Theorem 2.2 and suppose

$$e_1 = \cosh\psi h_1 + \sinh\psi h_2,\qquad e_2 = \sinh\psi h_1 + \cosh\psi h_2, \tag{3.20}$$

where e_2 is the TBC direction and $\psi = -(\alpha+\tilde{\alpha})/2$. By a direct calculation, we have

$$\omega^1 = \cosh\psi\,dx + \sinh(\psi+\tilde{\alpha})\,dy,\qquad \omega^2 = \sinh\psi\,dx + \cosh(\psi+\tilde{\alpha})\,dy,$$
$$\omega_1^2 = \eta_1^2 - d\psi,\qquad \omega_1^3 = \rho(-\sinh\psi\,dx + \cosh(\psi+\tilde{\alpha})\,dy),$$
$$\omega_2^3 = \rho(\cosh\psi\,dx - \sinh(\psi+\tilde{\alpha})\,dy). \tag{3.21}$$

Using (2.19), the fundamental forms of M^* are

$$I^* = (\omega^{*1})^2 - (\omega^{*2})^2 = \frac{\sinh l}{\sin\tau}(\omega_2^3)^2 - (\omega^2)^2 = \mathrm{d}x^2 + 2\sinh\alpha\,\mathrm{d}x\,\mathrm{d}y - \mathrm{d}y^2,$$
$$II^* = \omega^{*1}\omega_1^{*3} + \omega^{*2}\omega_2^{*3} = \frac{\sinh l}{\sin\tau}\omega_1^3\omega_2^3 + \frac{\sinh l}{\sin\tau}\omega^1\omega^2 = 2\rho\cosh\alpha\,\mathrm{d}x\,\mathrm{d}y.$$

By Lemma 3.7, we know (1) holds. Substituting (3.21) in (2.20), comparing the coefficients of $\mathrm{d}x$, $\mathrm{d}y$ in (2.20), we get the BT (3.19). Similarly one may prove Theorem 3.13. □

4. Loop group actions and BT between time-like surface and space-like surface

In the rest of the sections, we construct a local action of the group of rational maps from S^2 to $GL(2, C)$ on the space of solutions of "-1-flow" of the $sl(2, C)$-hierarchy and -1-flow associated to $SU(1, 1)/SO(1, 1)$. We show that the actions of simple elements give local BTs (Propositions 4.4 and 5.5). By suitable constraints, we describe the relations of loop group actions and BTs between time-like surface or space-like surface in S_1^3 (Theorems 4.5 and 5.6).

For the BT between space-like surfaces in S_1^3, actually it is the BT of sine-Gordon equation. The relation with loop group actions has been studied by Uhlenbeck and Terng [16,17]. In this section we shall consider the relation of loop group actions and the BT between time-like surface and space-like surface in S_1^3.

For Eqs. (3.9) and (3.11), we introduce complex coordinates of (x, y) plane

$$\eta = \frac{\sqrt{\rho^2 - 1}}{2}(x + \mathrm{i}y), \qquad \bar\eta = \frac{\sqrt{\rho^2 - 1}}{2}(x - \mathrm{i}y). \tag{4.1}$$

Then Eqs. (3.9) and (3.11) can be written as real sinh-Laplace equation and real sin-Laplace equation

$$\alpha_{\eta\bar\eta} = \sinh\alpha, \tag{4.2}$$

$$\tilde\alpha_{\eta\bar\eta} = \sin\tilde\alpha. \tag{4.3}$$

The BT (3.12) becomes

$$(\alpha - \mathrm{i}\tilde\alpha)_\eta = 2\bar\zeta\sinh\frac{\alpha + \mathrm{i}\tilde\alpha}{2}, \qquad (\alpha + \mathrm{i}\tilde\alpha)_{\bar\eta} = 2\zeta\sinh\frac{\alpha - \mathrm{i}\tilde\alpha}{2}.$$

where $\zeta = \sqrt{(\cos l + \mathrm{i}\sinh\tau)/(\cos l - \mathrm{i}\sinh\tau)} \in S^1$.

Proposition 4.1. *Suppose α is a solution of Eq.(4.2) and $\zeta \in S^1$. Then the following first-order system is solvable for $\tilde\alpha$*

$$(\alpha - \mathrm{i}\tilde\alpha)_\eta = 2\bar\zeta\sinh\frac{\alpha + \mathrm{i}\tilde\alpha}{2}, \qquad (\alpha + \mathrm{i}\tilde\alpha)_{\bar\eta} = 2\zeta\sinh\frac{\alpha - \mathrm{i}\tilde\alpha}{2}. \tag{4.4}$$

Moreover $\tilde\alpha$ is a solution of the sinh-Laplace equation (4.3).

Definition 4.2. If α is a solution of sinh-Gordon equation (4.2), then given any $c_0 \in R$ there is a unique solution $\tilde{\alpha}$ for Eq. (4.4) such that $\tilde{\alpha}(0,0) = c_0$ denoted by $\mathbf{B}_{\zeta,\mathbf{c}_0}(\alpha) = \tilde{\alpha}$ which is called the BT between real sinh-Laplace equation and real sin-Laplace equation.

Notice that Eqs. (4.2) and (4.3) can be obtained in the complex sinh-Laplace equation

$$\phi_{\eta\bar{\eta}} = \sinh\phi. \tag{4.5}$$

Its Lax pair is

$$\Phi^{-1}\Phi_{\bar{\eta}} = a\lambda + \begin{pmatrix} 0 & \dfrac{\phi_{\bar{\eta}}}{2} \\ \dfrac{\phi_{\bar{\eta}}}{2} & 0 \end{pmatrix} = A(\lambda),$$
$$\Phi^{-1}\Phi_{\eta} = \frac{1}{4}\begin{pmatrix} \cosh\phi & \sinh\phi \\ -\sinh\phi & -\cosh\phi \end{pmatrix}\lambda^{-1} = Q(\lambda), \tag{4.6}$$

where $a = \mathrm{diag}(1, -1)$.

In fact, when ϕ is real and $\phi = \alpha$, α satisfies Eq. (4.2). When ϕ is purely imaginary and $\phi = \mathrm{i}\tilde{\alpha}$ ($\tilde{\alpha}$ is real), $\tilde{\alpha}$ satisfies Eq. (4.3). From the Lax pair, we note that the complex sinh-Laplace equation (4.5) can be regarded as a "-1-flow" equation in the $sl(2, C)$-hierarchy defined by $b = (a/4) \in sl(2)_a^{\perp}$ is

$$u_\eta = [a, g^{-1}bg], \qquad g^{-1}g_{\bar{\eta}} = u, \qquad \lim_{\mathrm{Re}\eta\to-\infty} g(\eta, \bar{\eta}) = I, \tag{4.7}$$

where

$$u \in sl(2)_a^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} : q, r \in C \right\}.$$

Note that the "-1-flow" in the $sl(2, C)$-hierarchy (4.7) leaves the submanifold $q = r$ invariant and choose $q = r = \phi_\eta/2$, (4.7) is reduced to the complex sinh-Laplace equation (4.5). On this submanifold, the Lax pair satisfies the following reality condition:

$$\tau^{-1}A(-\lambda)\tau = A(\lambda), \qquad \tau^{-1}Q(-\lambda)\tau = Q(\lambda) \tag{4.8}$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the following we construct a local action of G_-^{CSHL} (defined later) on the space of solutions of the complex sinh-Gordon equations.

Definition 4.3. Let G_-^{CSHL} denote the group of rational maps $g : S^2 \to GL(2, C)$ such that (1) g is holomorphic at $\lambda = \infty$, and (2) there exists a rational function h such that gh satisfies the reality condition (4.8).

Given $v = (v_1, v_2)^t \in C^2$ and $k \in C$, we define a degree 1 rational map

$$g_{v,k}(\lambda) = \frac{a\lambda - kaB(v)aB(v)^{-1}}{\lambda - k}, \tag{4.9}$$

where $B(v) = (v, \tau^{-1}v)$ is non-singular. It is verified that $(\lambda - k)g_{v,k}(\lambda)$ satisfies the reality condition (4.8). So $g_{v,k}(\lambda) \in G_-^{\mathrm{CSHL}}$ and we call $g_{v,k}(\lambda)$ a simple element of G_-^{CSHL}. Analogous with [17], one may obtain the following proposition.

Proposition 4.4. *Let $v = (v_1, v_2)^t \in C^2$ and $k \in C$. Let $u(\eta, \bar\eta)$ be a local solution of the "−1-flow" Eq. (4.7) on the submanifold $q = r$ and Φ be the trivialization at $(0, 0)$, where $(\eta, \bar\eta) \in O_1$. Let $\tilde v(\eta, \bar\eta) = \Phi(\eta, \bar\eta, k)^{-1}(v)$ and $B(v) = (v, \tau^{-1}v)$. If $B(v)$ is non-singular, then there exists an open subset $O \subset O_1$ such that $B(\tilde v)$ is non-singular for all $(\eta, \bar\eta) \in O$. Moreover,*

(1) *$\tilde u = aua^{-1} + [\tilde Y, a]a^{-1}$ is a solution defined on O denoted by $\tilde u = g_{v,k}\#u$, where $\tilde Y(\eta, \bar\eta) = -kaB(\tilde v)aB(\tilde v)^{-1}$.*
(2) *$\tilde\Phi = g_{v,k}(\lambda)\Phi g_{\tilde v,k}^{-1}(\lambda)$ is the trivialization of $\tilde u$.*
(3) *$\tilde Y$ is a solution of*

$$(\tilde Y)_{\bar\eta} = \tilde Y u - (aua^{-1} + [\tilde Y, a]a^{-1})\tilde Y,$$
$$(\tilde Y)_\eta = ag^{-1}(u)bg(u) - \tilde Y g^{-1}(u)bg(u)\tilde Y^{-1}a, \qquad \tau^{-1}\tilde Y\tau = \tilde Y. \tag{4.10}$$

Proof. Note that (4.8) is also the reality condition for 2×2 KW-hierarchy. Hence analogous with Theorem 13.8 in [17], we may show $\tilde\Phi = g_{v,k}(\lambda)\Phi g_{\tilde v,k}^{-1}(\lambda)$ is holomorphic for $0 \neq \lambda \in C$ and the trivialization of $\tilde u$.

On the other hand, we know

$$a\lambda + \tilde u = \tilde\Phi^{-1}\tilde\Phi_{\bar\eta} = g_{\tilde v,k}\Phi^{-1}\Phi_{\bar\eta}g_{\tilde v,k}^{-1} - (g_{\tilde v,k})_{\bar\eta}g_{\tilde v,k}^{-1},$$

and $g_{\tilde v,k}(\lambda) = (a\lambda + \tilde Y)/(\lambda - k)$. Therefore

$$(a\lambda + \tilde u)(a\lambda + \tilde Y) = (a\lambda + \tilde Y)(a\lambda + u) - \tilde Y_{\bar\eta}.$$

Comparing the coefficient of λ^j for $j = 0, 1$, and we have

$$\tilde u = aua^{-1} + [\tilde Y, a]a^{-1}, \qquad \tilde Y_{\bar\eta} = \tilde Y u - \tilde u\tilde Y.$$

Substitute the first equation into the second equation, and we get

$$(\tilde Y)_{\bar\eta} = \tilde Y u - (aua^{-1} + [\tilde Y, a]a^{-1})\tilde Y.$$

Similarly we have

$$\lambda^{-1}g^{-1}(\tilde u)bg(\tilde u)(a\lambda + \tilde Y) = (a\lambda + \tilde Y)\lambda^{-1}g^{-1}(u)bg(u) - \tilde Y_\eta.$$

Comparing the coefficient of λ^j for $j = 0, -1$, and we obtain $(\tilde Y)_\eta = ag^{-1}(u)bg(u) - \tilde Y g^{-1}(u)bg(u)\tilde Y^{-1}a$. □

Now we describe the connection between the BT as in Definition 4.2 and the action of G_-^{CSHL} on the space of solutions of the complex sinh-Laplace equation (4.5). Given $0 \neq k \in \mathbf{C}$ and choose $\tilde{v} = (\cosh(f/2), \sinh(f/2))^t$, and then

$$\tilde{Y} = -k \begin{pmatrix} \cosh f & -\sinh f \\ -\sinh f & \cosh f \end{pmatrix}.$$

So the first-order system (4.10) for $\tilde{Y}$ becomes

$$f_{\bar{\eta}} = -\phi_{\bar{\eta}} + 2k \sinh f, \qquad f_\eta = \frac{1}{2k} \sinh(f + \phi). \tag{4.11}$$

Write

$$\tilde{u} = g_{v,k} \# u = \begin{pmatrix} 0 & \dfrac{\tilde{\phi}_{\bar{\eta}}}{2} \\ \dfrac{\tilde{\phi}_{\bar{\eta}}}{2} & 0. \end{pmatrix}.$$

But $\tilde{u} = aua^{-1} + [\tilde{Y}, a]a^{-1}$, hence we have $\tilde{\phi} = 2f + \phi$. Then we get

$$(\phi - \tilde{\phi})_\eta = \frac{1}{k} \sinh \frac{\phi + \tilde{\phi}}{2}, \qquad (\phi + \tilde{\phi})_{\bar{\eta}} = 4k \sinh \frac{\phi - \tilde{\phi}}{2}.$$

Note that if $\phi = \alpha$ is real, and taking $\mathrm{Re}\, f = -\phi/2$, then $\tilde{\phi} = 2\mathrm{i}\mathrm{Im}\, f$ is purely imaginary, we denote $\tilde{\phi} = \mathrm{i}\tilde{\alpha}$. So $\tilde{\alpha}$ satisfies (4.4) which is the BT of (3.12). Hence we have the following:

Theorem 4.5. *Let α is be solution of Eq.(4.2) and $c_0 > 0$. Set*

$$u = \begin{pmatrix} 0 & \dfrac{\alpha_{\bar{\eta}}}{2} \\ \dfrac{\alpha_{\bar{\eta}}}{2} & 0 \end{pmatrix}, \qquad f_0 = \tfrac{1}{2}(\mathrm{i}c_0 - \alpha(0,0)).$$

and $\tilde{v} = (\cosh(f_0/2), \sinh(f_0/2))^t$. Then

$$g_{v,\zeta/2} \# u = \begin{pmatrix} 0 & \dfrac{\mathrm{i}\tilde{\alpha}_{\bar{\eta}}}{2} \\ \dfrac{\mathrm{i}\tilde{\alpha}_{\bar{\eta}}}{2} & 0 \end{pmatrix},$$

where $\tilde{\alpha} = B_{\zeta,c_0}(\alpha)$ and $\zeta \in S^1$.

5. Loop group actions and BT between time-like surfaces

In this section we investigate the relation of loop group actions and BT between time-like surfaces with real principal curvatures. In Section 3 we have obtained that the Gauss equation of time-like surface with $K = 1 + \rho^2$ is

$$\alpha_{xx} - \alpha_{yy} = -(\rho^2 + 1)\sinh\alpha, \tag{5.1}$$

where (x, y) are Tchebyshev coordinates.

Note that if one makes a parameter transformation

$$x = \frac{1}{\sqrt{1+\rho^2}}(t - s), \qquad y = \frac{1}{\sqrt{1+\rho^2}}(t + s), \tag{5.2}$$

where (s, t) are called asymptotic coordinates. Then Eq. (5.1) becomes sinh-Gordon equation:

$$\alpha_{st} = \sinh\alpha. \tag{5.3}$$

Hence the BT between time-like surfaces is the BT of Eq. (5.3). A direct calculation shows that system (3.16) (or (3.17)) becomes

$$(\alpha - \tilde{\alpha})_s = 4\zeta \sinh\frac{\alpha + \tilde{\alpha}}{2}, \qquad (\alpha + \tilde{\alpha})_t = \frac{1}{\zeta}\sinh\frac{\alpha - \tilde{\alpha}}{2},$$

where

$$\zeta = \frac{1}{2}\sqrt{\frac{\cosh\tau + \cos l}{\cosh\tau - \cos l}} \quad \left(\text{or } \frac{1}{2}\sqrt{\frac{\cosh l + \cos\tau}{\cosh l - \cos\tau}}\right).$$

So we have the following:

Proposition 5.1. *Suppose α is a solution of Eq.(5.3) and $\zeta \neq 0$ is a real number. Then the following first-order system is solvable for $\tilde{\alpha}$*

$$(\alpha - \tilde{\alpha})_s = 4\zeta \sinh\frac{\alpha + \tilde{\alpha}}{2}, \qquad (\alpha + \tilde{\alpha})_t = \frac{1}{\zeta}\sinh\frac{\alpha - \tilde{\alpha}}{2}. \tag{5.4}$$

Moreover, $\tilde{\alpha}$ is a solution of Eq.(5.3).

Definition 5.2. If α is a solution of sinh-Gordon equation (5.3), then given any $c_0 \in R$ there is a unique solution $\tilde{\alpha}$ for Eq. (5.4) such that $\tilde{\alpha}(0, 0) = c_0$ denoted by $\mathbf{B}_{\zeta,\mathbf{c_0}}(\alpha) = \tilde{\alpha}$ which is called the BT for sinh-Gordon equation.

For sinh-Gordon equation (5.3), its Lax pair is

$$\Phi^{-1}\Phi_s = a\lambda + \begin{pmatrix} 0 & \frac{\alpha_s}{2} \\ \frac{\alpha_s}{2} & 0 \end{pmatrix}, \qquad \Phi^{-1}\Phi_t = \frac{i}{4}\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ -\sinh\alpha & -\cosh\alpha \end{pmatrix}\lambda^{-1} \tag{5.5}$$

where $a = \mathrm{diag}(i, -i)$. From Lax pair, we find the sinh-Gordon equation (5.3) could be derived from the -1-flow associated to the Lorentzian symmetric space $SU(1, 1)/SO(1, 1)$. The -1-flow equation [14,15] associated to $SU(1, 1)/SO(1, 1)$ defined by $b = -a/4$ is

$$u_t = [a, g^{-1}bg], \qquad g^{-1}g_s = u, \qquad \lim_{s\to-\infty} g(s, t) = I, \tag{5.6}$$

where $g : R^2 \to SO(1,1)$,

$$u \in su(1,1)^{\perp}_{a,\sigma_1} = \left\{ \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} : v \in R \right\}.$$

Then the -1-flow equation (5.6) for

$$u = \begin{pmatrix} 0 & \frac{\alpha_s}{2} \\ \frac{\alpha_s}{2} & 0 \end{pmatrix}$$

is the sinh-Gordon equation (5.3). In the following, we also construct a simple element action of $G_-^{m,\sigma}$ on the space of solutions of the sinh-Gordon equation (5.3).

Definition 5.3. Let $\mathcal{O}_\infty$ denote a neighbourhood of ∞ in S^2. $G_-^{m,\sigma}$ denotes the group of rational map $g : \mathcal{O}_\infty \cap C \to GL(2, C)$ such that (1) g is a holomorphic map and $g(\infty) = I$; (2) g satisfies the $SU(1,1)$ reality condition: $g(\bar{\lambda})^* J g(\lambda) = J$; (3) $\sigma(g(-\lambda)) = g(\lambda)$, where σ is an involution σ on $SU(1,1)$ denoted by $\sigma(g) = J(g^t)^{-1}J^{-1}$, $J = \mathrm{diag}(1, -1)$.

Let $z \in C$ and π be J-projection of C^2 onto a complex linear space, i.e., $\pi^{*J} = \pi$, where $\pi^{*J} = J^{-1}\pi^* J$. Set

$$g_{z,\pi} = \pi + \frac{\lambda - z}{\lambda - \bar{z}}(I - \pi). \tag{5.7}$$

It is easy to check that $I - \pi$ is a J-projection and $g_{z,\pi}(\bar{\lambda})^* J g_{z,\pi}(\lambda) = J$. Hence we have

Proposition 5.4. *$g_{z,\pi} \in G_-^{m,\sigma}$ if and only if $z = -\bar{z}$ and $\bar{\pi} = \pi$.*

Proof. $g_{z,\pi} \in G_-^{m,\sigma} \Leftrightarrow \sigma(g_{z,\pi}(-\lambda)) = g_{z,\pi}(\lambda)$

$\Leftrightarrow J(g^t_{z,\pi}(-\lambda))^{-1}J^{-1} = Jg_{z,\pi}(\bar{\lambda})^* J^{-1}$, using $SU(1,1)$ reality condition

$\Leftrightarrow g_{z,\pi}(-\lambda) = \overline{g_{z,\pi}(\bar{\lambda})} \Leftrightarrow z = -\bar{z}, \bar{\pi} = \pi.$ □

Analogous with Proposition 4.4, we could obtain the following proposition.

Proposition 5.5. *Let $u : \mathcal{O}_\infty \to su(1,1)^{\perp}_{a,\sigma_*}$ be a local solution of the -1-flow equation (5.6), Φ the trivialization and $g_{z,\pi}$ a simple element in $G_-^{m,\sigma}$. V_1 and V_2 denote the image of J-projections π and $I - \pi$, respectively. Then there exists an open subset $\mathcal{O} \subset \mathcal{O}_\infty$ such that $\tilde{V}_1(x,t) \cap \tilde{V}_2(x,t) = 0$ for $(x,t) \in \mathcal{O}$. Moreover, let $\tilde{\pi}(x,t)$ denote the J-projection onto $\tilde{V}_1(x,t)$ with respect to $C^2 = \tilde{V}_1(x,t) \oplus \tilde{V}_2(x,t)$. Then*

(1) *$\tilde{u} : \mathcal{O} \to su(1,1)^{\perp}_{a,\sigma}$ defined by $\tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a]$ is a solution of the -1-flow equation (5.6) denoted by $\tilde{u} = g_{z,\pi} \# u$,*
(2) *$\tilde{\Phi} = g_{z,\pi}(\lambda)\Phi g^{-1}_{z,\tilde{\pi}}(\lambda)$ is the trivialization of $\tilde{u}$,*

(3) $\tilde{\pi}$ *is a solution of*

$$(\tilde{\pi})_s = [\tilde{\pi}, az + u] + (z - \bar{z})[\tilde{\pi}, a]\tilde{\pi},$$
$$(\tilde{\pi})_t = \frac{1}{|z|^2}((z - \bar{z})\tilde{\pi} g^{-1}(u)bg(u)\tilde{\pi} - zg^{-1}(u)bg(u)\tilde{\pi} + \bar{z}\tilde{\pi} g(u)^{-1}bg(u)),$$
$$\tilde{\pi}^{*J} = \tilde{\pi}, \qquad \tilde{\pi}^2 = \tilde{\pi}, \qquad \tilde{\pi}(0,0) = \pi. \tag{5.8}$$

Now we relate the BT as in Definition 5.2 and the action of $G_{-}^{m,\sigma}$ on the space of solutions of the sinh-Gordon equation (5.3). Given $0 \neq \zeta \in R$ and $\tilde{\pi}^* J = J\tilde{\pi} = \tilde{\pi}^t J$, then by Proposition 5.4, $g_{i\zeta,\tilde{\pi}} \in G_{-}^{m,\sigma}$ and is called a simple element of $G_{-}^{m,\sigma}$. Hence $\tilde{\pi}$ is a J-projection of C^2 onto $(\cosh(f/2), \sinh(f/2))$ for some function $f(s,t)$, i.e.

$$\tilde{\pi} = \begin{pmatrix} \cosh^2 \dfrac{f}{2} & -\cosh\dfrac{f}{2}\sinh\dfrac{f}{2} \\ \cosh\dfrac{f}{2}\sinh\dfrac{f}{2} & -\sinh^2\dfrac{f}{2} \end{pmatrix}. \tag{5.9}$$

So the first-order system (5.8) for $\tilde{\pi}$ becomes

$$f_s = -\alpha_s + 2\zeta\sinh f, \qquad f_t = \frac{1}{2\zeta}\sinh(f + \alpha). \tag{5.10}$$

Set

$$\tilde{u} = g_{i\zeta,\tilde{\pi}} \# u \begin{pmatrix} 0 & \dfrac{\tilde{\alpha}_s}{2} \\ \dfrac{\tilde{\alpha}_s}{2} & 0 \end{pmatrix}.$$

By Proposition 5.5, $\tilde{u} = u + 2i\zeta[\tilde{\pi}, a]$. Hence we have $\tilde{\alpha} = -2f - \alpha$. Then we get

$$(\alpha - \tilde{\alpha})_s = 4\zeta\sinh\frac{\alpha + \tilde{\alpha}}{2}, \qquad (\alpha + \tilde{\alpha})_t = \frac{1}{\zeta}\sinh\frac{\alpha - \tilde{\alpha}}{2},$$

which is the BT for the sinh-Gordon equation (5.3). So we have the following:

Theorem 5.6. *Let* α *is a solution of the sinh-Gordon equation*(5.3) *and* $c_0 > 0$. *Set*

$$u = \begin{pmatrix} 0 & \dfrac{\alpha_s}{2} \\ \dfrac{\alpha_s}{2} & 0 \end{pmatrix}, \qquad f_0 = \tfrac{1}{2}(\alpha(0,0) + c_0).$$

and π *is the J-projection onto the complex linear subspace spanned by* $(\cosh(f_0/2), \sinh(f_0/2))$. *Then*

$$g_{i\zeta,\pi} \# u = \begin{pmatrix} 0 & \dfrac{\tilde{\alpha}_s}{2} \\ \dfrac{\tilde{\alpha}_s}{2} & 0 \end{pmatrix},$$

where $\tilde{\alpha} = B_{\zeta,c_0}(\alpha)$ *and* $0 \neq \zeta \in R$.

Remark 5.7. For solving the equations for BTs, the Darboux transformation method has been used to obtain the explicit formulas of possible solutions [13,18,20]. Loop group action method is also an effective method. But we need Bianchi permutability formulas, similar to [17], which could be derived by factoring quadratic elements in the rational loop group $G_-^{\mathrm{CSHL}}(G_-^{m,\sigma})$ in two ways as a product of two simple elements.

6. Bäcklund theorems in H_1^3

In this section, we shall generalize the classical Bäcklund theorem in H_1^3.

Let $R^{2,2}$ denote the four-dimensional Lorentz space endowed with linear coordinates (X_0, X_1, X_2, X_3) and the scalar product $\langle , \rangle$ given by $-X_0^2 - X_1^2 + X_2^2 + X_3^2$. The three-dimensional anti-de Sitter space H_1^3 of constant sectional curvature -1 is defined as the following hyper-quadric in $R^{2,2}$

$$H_1^3 = \{X \in R^{2,2} | \langle X, X\rangle = -1\}.$$

In H_1^3 we may also introduce the corresponding four kinds of BCs denoted by $L_j (1 \leq j \leq 4)$,where L_1, L_2 and L_3 are SBCs and L_4 is a TBC. In the following, all calculations are parallel to the above sections.

Theorem 6.1. *Let M and M^* be two immersed surfaces in H_1^3. Let $L_j : M \to M^*$ $(1 \leq j \leq 4)$ in H_1^3 be one of the above BCs as in Definition 2.1. Then M and M^* have the same constant Gaussian curvature K, where in* (1) $K = -1 - (\cosh^2\tau/\sinh^2 l)$ *and* $c = \sinh\tau$; (2) $K = -1 + (\sinh^2\tau/\sinh^2 l)$ *and* $c = -\cosh\tau$; (3) $K = -1 + (\sinh^2\tau/\sinh^2 l)$ *and* $c = \cosh\tau$; *and* (4) $K = -1 + (\sin^2\tau/\sin^2 l)$ *and* $c = \cos\tau$.

Theorem 6.2. *Suppose M is an immersed time-like surface with $K = -1 - (\cosh^2\tau/\sinh^2 l)$ in H_1^3. Given any unit space-like vector $v_0 \in T_{p_0}M$, $p_0 \in M$, which is not a principal direction. Then there exist a unique space-like surface M^* with K and the above SBC L_1 such that $L_1(p_0) = \cosh lp_0 + \sinh lv_0$, where the BT is $\omega^2 \cosh l + \omega_1^2 \sinh l = \omega_1^3 \sinh l \tanh\tau$.*

Theorem 6.3. *Suppose M is an immersed space-like* (*or time-like*) *surface with $K = -1 + (\sinh^2\tau/\sinh^2 l)$ in H_1^3. Given any unit space-like vector $v_0 \in T_{p_0}M$, $p_0 \in M$, which is not a principal direction. Then there exist a unique space-like* (*or time-like*) *surface M^* with K and the above SBC L_2* (*or L_3*) *such that $L_2(p_0)$* (*or $L_3(p_0)$*) $= \cosh lp_0 + \sinh lv_0$, *where the BT is $\omega^2 \cosh l + \omega_1^2 \sinh l = \omega_1^3 \sinh l \coth\tau$.*

Theorem 6.4. *Suppose M is an immersed time-like surface with $K = -1 + (\sin^2\tau/\sin^2 l)$ in H_1^3. Given any unit time-like vector $v_0 \in T_{p_0}M$, $p_0 \in M$, which is not a principal direction. Then there exist a unique time-like surface M^* with K and the above TBC L_4 such that $L_4(p_0) = \cos lp_0 + \sin lv_0$, where the BT is $\omega^1 \cos l + \omega_2^1 \sin l = \omega_2^3 \sin l \cot\tau$.*

Remark 6.5. In H_1^3, we also set up the Tchebyshev coordinates and discuss the explicit forms of BTs. For time-like surface with $K = -1 + \rho^2$ and imaginary principal curvatures, the Gauss equation is $\alpha_{xx} - \alpha_{yy} = (1 - \rho^2) \sinh \alpha$. Especially when time-like surface is flat, i.e., $K = 0$, it is a wave equation.

Acknowledgements

The authors want to thank the referee for pointing out the missed case of time-like surfaces of positive constant curvature with imaginary principal curvatures and for other valuable remarks which improve the presentation of the paper. This work was supported by the NSFC 19725104 and the Chinese National Basic Research Project "Nonlinear Science".

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